Alternating Deadheading in Bus Route Operations

PETER G. FURTH

Northeastern University, Boston, Massachusetts

"Alternating deadheading" is an operating strategy for urban bus routes that have a directional imbalance in passenger demand in which some of the vehicles operating on a route deadhead (return empty) in the reverse direction while others return in service. By reducing average cycle time, deadheading can reduce the number of buses needed to serve a route. A formula is developed for the number of buses needed to meet a regular alternating deadheading schedule. Design procedures are then presented for finding the alternating deadheading schedule that will minimize the number of vehicles needed subject to the usual operating constraints, and for minimizing total wait time for a given fleet size. Application to a major local bus route demonstrates the potential of this strategy to reduce fleet size within typical scheduling constraints.

During peak periods, bus routes commonly exhibit a considerable directional imbalance in passenger flows. Matching supply to demand, then, would suggest offering more frequent service in one direction than another. In order to circulate vehicles on the route, some vehicles will have to deadhead, or return empty, in the reverse direction, while other vehicles return in service. This strategy is called "alternating deadheading." By deadheading, vehicles become free to use the fastest path available to complete the reverse direction trip and thereby become available sooner to make another peak direction trip.

Deadheading vehicles on express routes is a common practice in the transit industry. However, alternating deadheading is probably not practiced on local routes as much as it could be to improve operating efficiency. Perhaps one of the reasons that it is not widely used is because of the requirement, assumed throughout this paper, that the trips that do not deadhead in the reverse direction should be evenly spaced in order to maintain balanced loads and acceptable wait times. At the same time,

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peak direction trips must also be evenly spaced. These requirements complicate the scheduling of alternating deadheading service.

In analyzing the deadheading strategy, this paper first considers the problem of finding the fleet size needed to meet a given alternating deadheading schedule. Next, it considers the design problem of constructing the alternating deadheading schedule that minimizes the required fleet size subject to level of service constraints. Another problem examined is finding the alternating deadheading schedule that minimizes wait time for a given fleet size. The problem of minimizing a sum of passenger and operator cost is also treated, using a continuous approximation. Finally, an application demonstrates how alternating deadheading can reduce fleet requirements on a heavy demand local bus route.

The strategy of systematic alternating deadheading on a local route does not appear in the literature. In the general field of vehicle scheduling, DANTZIG AND FULKERSON[1] have shown that finding the minimum number of vehicles to meet a given schedule is a linear optimization problem. They originally suggested solving it by the simplex method. Later, by using Dilworth's chain decomposition theorem, FORD AND FULKERSON[2] showed that it could be solved more efficiently with a maximum flow algorithm. These classical approaches both require an iterative search. More recently, CEDER AND STERN[3] have used the deficit function to compute the fleet size needed by a given schedule. This approach requires only a computation of the cumulative departures and arrivals over time at each terminal in the system. Based on this approach, they have developed iterative heuristics to insert deadheading trips into a given schedule in order to minimize the fleet size required. This approach, like the classical approaches, is applicable to a general schedule involving many terminals and arbitrarily scheduled trips. In contrast, the alternating deadheading problem that this paper addresses is much narrower in that it involves only two terminals between which vehicles circulate and because in-service trips in each direction have regular headways, although the headways differ by direction. Because of these restrictions, it is possible to determine in closed form the fleet size required to meet a given alternating deadheading schedule. With this result as a base, algorithms are then developed for the design of optimal alternating deadheading schedules where service headways are themselves control variables.

1. MINIMUM NUMBER OF VEHICLES TO MEET A GIVEN ALTERNATING DEADHEADING SCHEDULE

FOR THE SAKE of simplicity in presentation, we will assume throughout this paper that the peak direction is inbound. We also assume that run time is a deterministic function of headway, and that, in order to maintain

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The following notation is u

h_A = headway inbound
h_B = service headway
t_A = run time inbounc
t_B = run time outbou layover)
t_D = run time outbour layover)
H_A = maximum allowa
H_B = maximum allowa
$r = h_B/h_A$
$R = H_B/H_A$
$\langle x \rangle$ = x rounded up
n_A = number of vehicl
n_B = additional numb
$N = n_A + n_B$ = requir

The maximum service constraints on the headwa

Our approach is best ir using alternating deadheac

H_A = 4 min
H_B = 13 min
t_A = 30 min when h_A
t_B = 25 min when h_B
t_B = 29 min when h_B
t_D = 11 min.

Conventional local service 4⟩ = 14 vehicles.

Suppose that an altern which the headways of in- 4 min and h_B = 13 min. T 1/4, the outbound service (1/4 − 1/13) = 0.173. If minimum necessary) is b hours needed per hour of = 11.63. Rounding up, the these headways would be in the schedule. However insufficient to meet the r and that achieving these vehicles, as shown below.

balanced loads, in-service trips must be scheduled at regular intervals. The following notation is used:

h_A = headway inbound
h_B = service headway outbound
t_A = run time inbound (including minimum layover)
t_B = run time outbound for in-service trips (including minimum layover)
t_D = run time outbound for deadheading trips (including minimum layover)
H_A = maximum allowable inbound headway
H_B = maximum allowable outbound service headway
$r = h_B/h_A$
$R = H_B/H_A$
$\langle x \rangle$ = x rounded up
n_A = number of vehicles needed if all deadhead
n_B = additional number of vehicles needed
$N = n_A + n_B$ = required fleet size.

The maximum service headways, H_A and H_B, are determined by constraints on the headway (policy headway) and the peak load.

Our approach is best introduced with an example. Suppose a route using alternating deadheading has the following parameters:

H_A = 4 min
H_B = 13 min
t_A = 30 min when $h_A = 4$
t_B = 25 min when $h_B = 4$
t_B = 29 min when $h_B = 13$
t_D = 11 min.

Conventional local service, at 4-min headways, would require $\langle(30 + 25)/4\rangle = 14$ vehicles.

Suppose that an alternating deadheading schedule is constructed in which the headways of in-service trips are set at their maxima, i.e. h_A = 4 min and h_B = 13 min. The inbound frequency (in trips/min) would be 1/4, the outbound service frequency 1/13, and the deadheading frequency $(1/4 - 1/13) = 0.173$. If no slack time (additional layover beyond the minimum necessary) is built into the schedule, the number of vehicle-hours needed per hour of operation would be $30/4 + 29/13 + (11)(0.173) = 11.63$. Rounding up, the minimum number of vehicles needed to achieve these headways would be 12, implying the inclusion of some slack time in the schedule. However, it turns out that this amount of slack time is insufficient to meet the restriction of evenly spaced service departures, and that achieving these headways will actually require more than 12 vehicles, as shown below.

The time-space network shown in Figure 1 illustrates the alternating deadheading schedule that results when the evenly spaced departures restriction is imposed. The left side of the network represents the uptown terminal; the right side, the downtown terminal. Each node represents a point in time and space. The time axis moves downward along the page. Each arc (i, j) represents a possible vehicle trajectory. The network contains five types of arcs:

1. Inbound arcs, representing inbound trips, are directed from left to right at intervals of h_A, requiring t_A minutes;
2. Outbound arcs, representing outbound in-service trips, are directed from right to left at intervals of h_B, requiring t_B minutes;
3. Deadhead arcs, representing potential deadheading trips, are also directed from right to left, departing immediately after every inbound arrival, requiring t_D minutes;
4. Wait arcs, representing extra layover (slack time) at a terminal, are directed from every node to the node immediately below it on each side of the network;
5. Garage arcs, representing bus movements out of and into the garage, are directed from the source node to each terminal and from each terminal to the sink node.

The schedule pictured in Figure 1 has been constructed efficiently in that its deadhead trips all depart immediately following an inbound arrival, and that at least one (the first) outbound service departure immediately follows an inbound arrival. The remainder of the schedule is dictated by the inbound and outbound service headways.

Each arc (i, j) has a minimum required flow w_{ij} and a maximum allowed flow u_{ij}. For the service arcs (the inbound and outbound arcs), the minimum required flow is 1 and the allowed flow is infinity. Wait arcs and deadhead arcs have required flows of 0 and allowed flows of infinity. A flow N enters the source node and leaves the sink node, representing the total number of vehicles used on the route. Thus, the problem of finding the minimum number of vehicles required to meet this alternating deadheading schedule is equivalent to finding the minimum flow N in this network that satisfies the flow requirements of every arc. This problem is solved using the "min-flow max-cut" theorem (analogous to the better known "max-flow min-cut" theorem), which states that the minimum feasible flow in a network with a single source and single sink is equal to the flow requirement of the cut with the greatest flow requirement.[4] In this context, a cut (X, Y) is defined as an imaginary curve that separates the set of nodes in the network into two mutually exclusive and collectively exhaustive subsets, X and Y, such that subset X contains the source node and subset Y contains the sink node. The

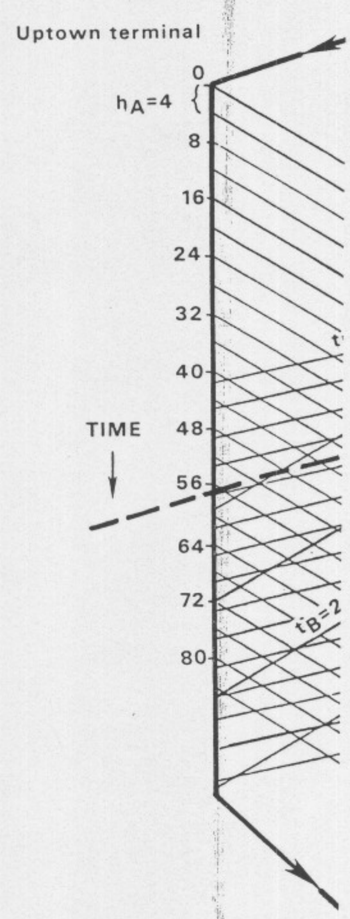

Fig. 1. Time-

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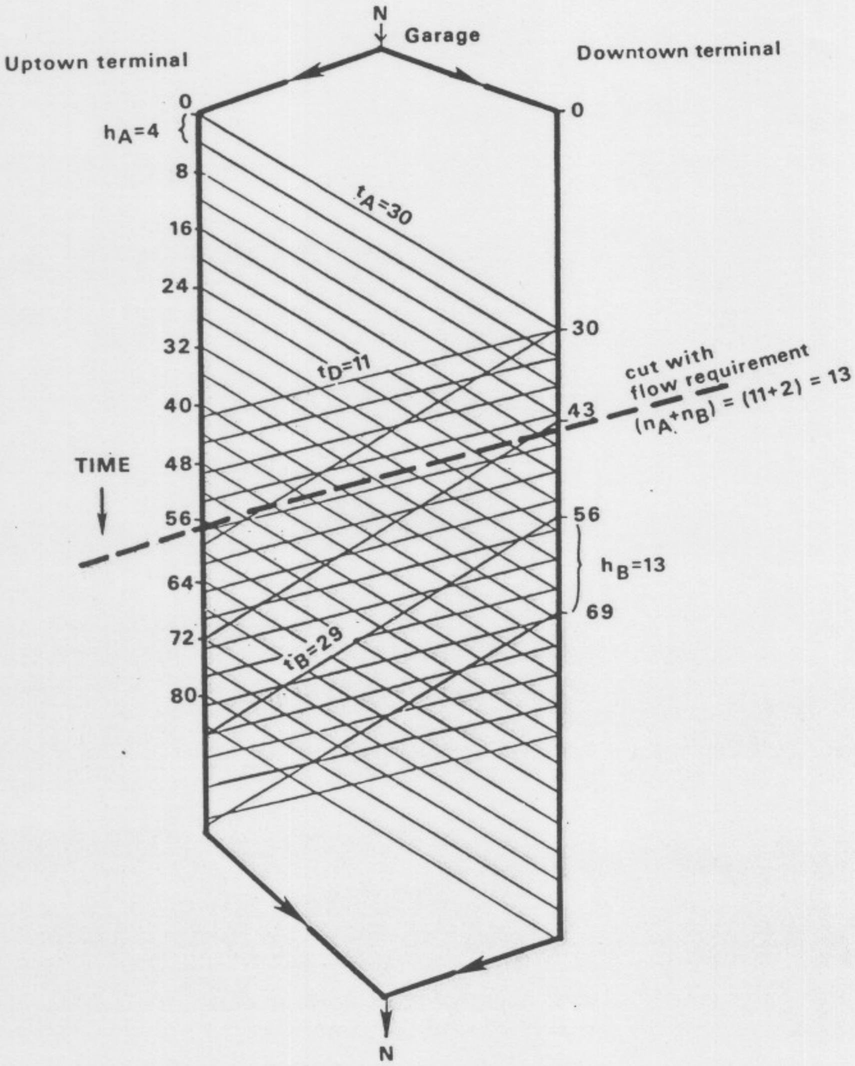

Fig. 1. Time-space diagram of deadheading route.

flow requirement of cut (X, Y), denoted $R(X, Y)$, is defined as

$$R(X,\ Y) = \textstyle\sum_{\text{all}\, i\in X} \sum_{\text{all}\, j\in Y} w_{ij} - \sum_{\text{all}\, i\in Y} \sum_{\text{all}\, j\in X} u_{ij}. \qquad (1)$$

The first term of Equation 1 implies that the flow requirement of a cut increases by 1 for each service arc (i, j) it intersects so as to leave node i in set X and node j in set Y. The second term implies that any cut that intersects an arc (i, j) so as to leave node i in set Y and node j in set X

has a flow requirement of $-\infty$. Figure 1 illustrates a cut (which happens to be a maximum flow cut) with a flow requirement of 13.

Our goal now is to find a maximum flow cut ("maximum cut"). First, we will show that the maximum cut will be a "single frame cut," where a single frame cut is defined as a cut that lies entirely within the "frame" lying between two adjacent deadhead arcs, and that intersects every inbound arc that enters that frame. To prove this, construct a single frame cut. Fix the left end of the cut and imagine rotating the right end. The right end cannot rotate up to another frame without making the cut intersect a deadhead arc in such a way that the flow requirement becomes $-\infty$. If the right end rotates down to the next lower frame, the cut will lose one inbound arc and will gain at most one outbound arc, since the width of a frame is h_A and outbound arcs are spaced at intervals of h_B, and $h_B > h_A$ by definition. Similarly, as the cut continues to rotate down it will lose one inbound arc and gain at most one outbound arc every time it enters a new frame. Therefore, the cut requirement cannot be increased by having a cut lie in more than one frame. By similar argument, it can be shown that if a cut that lies in a single frame avoids intersecting an inbound arc that partially traverses that frame, it can by doing so intersect at most one additional outbound arc, and thus cannot have a net increase in its flow requirement.

The flow requirement of a single frame cut is the sum of the number of inbound arcs traversed by that cut and the number of outbound arcs traversed by that cut. Every single frame cut, apart from those near the source and sink, intersects a constant number of inbound arcs n_A, given by

$$n_A = \langle (t_A + t_D)/h_A \rangle. \tag{2}$$

Note that n_A is the number of vehicles that would be required to serve the route if every trip deadheaded. In our example, $n_A = \langle (30 + 11)/4 \rangle = 11$, as the single frame cut in Figure 1 confirms.

To find the maximum cut, then, it is necessary to locate the single frame cut that traverses the greatest number of outbound arcs. Consider the frame whose lower deadhead arc departs from the right side at time t_0. The earliest inbound arc intersected by that cut is the inbound arc that arrives at time t_0; this arc departs from the left side at time $t_0 - t_A$. Then since the cut intersects n_A inbound arcs, the latest inbound arc intersected by the cut departs at time $t_0 - t_A + h_A(n_A - 1)$. The earliest outbound arc intersected by the cut must therefore arrive at the left side after this time, and hence it must depart from the right side after time $t_0 - t_A + h_A(n_A - 1) - t_B$, which is an interval of length $(\pi + h_A)$ before time t_0, where π is the "effective deadhead premium" given by

$$\pi = t_A + t_B - h_A n_A. \tag{3}$$

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and thus

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Then since the latest outbound arc intersected by the cut must depart from the right side before time t_0, the outbound arcs intersected by a single frame cut will be those that depart within the interval of length $(\pi + h_A)$ preceding an inbound arrival.

To find the maximum cut, then, we want to find the maximum number of outbound departures that can occur in the interval of length $(\pi + h_A)$ preceding an inbound arrival. This quantity depends in part on the phasing of inbound and outbound departures. The optimal phasing (which minimizes this maximum) is to have an outbound departure immediately follow an inbound arrival at (at least) one point in the schedule. Let time 0 be such a time of coincidence.

In order to find the single frame cut that intersects the greatest number of outbound arcs, it is clear that we need only consider frames that contain an outbound departure. (By argument similar to those given previously, if another frame is chosen, one outbound arc will be lost, and no more than one gained.) Consider then frame k, the frame during which outbound arc k, the kth outbound departure after time 0, departs. Also consider cut k, the single frame cut lying in this frame. Let the first inbound arrival that follows the departure of outbound arc k be called inbound arrival k. Since inbound arrivals follow time 0 at intervals of h_A while outbound departures follow time 0 at intervals of h_B, outbound departure k will precede inbound arrival k by the interval of length $I(k)$, given by

$$I(k) = h_A[1 - \mathrm{mod}(kr)]$$

where r is the headway ratio h_B/h_A and where $\mathrm{mod}(x)$ is the fractional component of x.

Using this result, then, the number of outbound departures that will occur in the interval of length $(\pi + h_A)$ preceding inbound arrival k is $\langle [\pi + h_A - I(k)]/h_B \rangle$. Substituting for $I(k)$, and taking k as a control variable, the maximum number of outbound arcs intersected by a single frame cut is

$$n_B = \max_k \langle (\pi + h_A \mathrm{mod}(kr))/h_B \rangle.$$

This maximization is equivalent to maximizing $\mathrm{mod}(kr)$. If we express r as the reduced ratio x/y (i.e. x and y are integers that have no common factor other than 1), then this maximum is

$$g(r) = \max_{k=1,2,\ldots} \mathrm{mod}(kr) = (y-1)/y \qquad (4)$$

and thus

$$n_B = \langle (\pi + g(r)h_A)/rh_A \rangle. \qquad (5)$$

Applying these formulas to our example, we have

$$n_A = 11$$

$$\pi = 30 + 29 - (4)(11) = 15 \text{ min}$$

$$r = 13/4 = 3.25$$

$$g(r) = (4 - 1)/4 = 0.75$$

$$n_B = \langle[15 + (0.75)(4)]/13\rangle = 2$$

$$N = 11 + 2 = 13.$$

2. MINIMUM NUMBER OF VEHICLES UNDER MAXIMUM HEADWAY CONSTRAINTS

CONTRARY to what might be expected, setting the service headways at their upper bounds does not necessarily yield the minimum vehicle requirement. The problem of choosing the service headways that will minimize the fleet requirement subject to the headway upper bound constraints is studied in light of two operator policies. Some operators may prefer that the service headway ratio r be an integer. This integer policy makes operations more simple, for when r is an integer, the first of every r trips will return in service and the remaining trips will deadhead. Other operators may allow noninteger headway ratios, which lead to more complex schedules.

We also make two very general assumptions. The first is that run time is a nondecreasing function of headway. The second is that run time is less than unit elastic with respect to service headway, i.e.

$$(\partial t_A/\partial h_A)(h_A/t_A) < 1 \quad \text{and} \quad (\partial t_B/\partial h_B)(h_B/t_B) < 1.$$

(Since at greater headways each vehicle carries more passengers, entailing more stops and longer dwell times, we expect the first assumption to hold; and violation of the second assumption would imply that the number of vehicle hours of operation needed on a route could be reduced by reducing the headway, a possibility that in practice is virtually inconceivable.)

When r is restricted to integer values, then $g(r) = 0$, and the following results can be derived from seeking to minimize N. These results make use of the ratio of the headway upper bounds $R = H_A/H_B$ and the integers obtained by rounding R up and down, R^+ and R^-.

1. Given r, N decreases with h_A, so therefore either $h_A = H_A$, or $h_B = H_B$;
2. Given h_A, N decreases with r, so when $h_A = H_A$, $r = R^-$;
3. Given h_B, N increases with r, so when $h_B = H_B$, $r = R^+$.

Therefore the optimal
H_B, $r = R^+$].

Thus, in our example,
parameters (h_A, h_B, r) wh
12, 3) or (3.25, 13, 4). If w
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1. Given r, $g(r)$ is fixed a
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3. When $h_B = H_B$, the opt

For the case $h_A = H_A$,
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below R^+, the function $g(r)$
Note that $g(r) = z_1$ at all
is an integer. Eliminating
values of $g(r)$ are bounded
which intersects $g(r)$ at all
is an integer that is prime
$- 1 + \text{mod}(r))/t$, $t = 3, 4,$
illustrated in Figure 2, whe
can also approximate $\pi(h_A$
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Algorithm 1

1. Initialize. Set $h_A = H_A$
   Using h_A and r, comput
2. With r variable, n_B is bo

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   where $k_0 = \pi_1/h_A$ and k
   may be skipped if t has
3. Let $r^- = k_1/(n_B - 1 - k_0$
   $= n_B - 1.)$ If $r^- > r^+$, ST

Therefore the optimal solution is either $[h_A = H_A, r = R^-]$, or $[h_B = H_B, r = R^+]$.

Thus, in our example, $R = 13/4 = 3.25$, and the optimal schedule parameters (h_A, h_B, r) when r is restricted to integer values are either (4, 12, 3) or (3.25, 13, 4). If we assume t_A and t_B are linear in this range with slopes of 1.0 and 0.44, respectively, the first solution yields $N = 13$ and the second $N = 15$; hence the first is the optimal integer headway ratio solution.

When r is not restricted to integer values, the following results can be found.

1. Given r, $g(r)$ is fixed and N decreases with h_A, so therefore either $h_A = H_A$ or $h_B = H_B$;
2. When $h_A = H_A$, the optimal r lies between R^- and R;
3. When $h_B = H_B$, the optimal r lies between R and R^+.

For the case $h_A = H_A$, we have to search for the optimal r over the range R^- to R. It can be observed that for r within the range R^- to just below R^+, the function $g(r)$ is bounded from below by the line $z_1 = \text{mod}(r)$. Note that $g(r) = z_1$ at all points for which $\text{mod}(r) = (y - 1)/y$, where y is an integer. Eliminating these points of intersection, the remaining values of $g(r)$ are bounded from below by the line $z_2 = (1 + \text{mod}(r))/2$, which intersects $g(r)$ at all points for which $\text{mod}(r) = (y - 2)/y$, where y is an integer that is prime with respect to 2. Successive envelopes $z_t = (t - 1 + \text{mod}(r))/t$, $t = 3, 4, \ldots$, may be constructed. These envelopes are illustrated in Figure 2, where their intersections with $g(r)$ are circled. We can also approximate $\pi(h_A, r)$ as a linear function $\pi = \pi_0 + \pi_1 r$ over the small range of r in which we are interested. With this approximation, n_B is bounded from below by a family of inverse linear functions, suggesting the following algorithm for finding the feasible value of r that minimizes n_B, given that $h_A = H_A$ and that r may be noninteger.

Algorithm 1

1. Initialize. Set $h_A = H_A$, $r = r^* = R^-$, $r^+ = R$, $t = \langle 1/\text{mod}(r) \rangle - 1$. Using h_A and r, compute n_A and n_B.
2. With r variable, n_B is bounded from below by the function

$$(\pi_0 + \pi_1 r + h_A[t - 1 + \text{mod}(r)]/t)/rh_A \tag{6}$$

which can be reduced to the function

$$k_0 + k_1/r \tag{7}$$

where $k_0 = \pi_1/h_A$ and $k_1 = \pi_0/h_A + [t - 1 + \text{mod}(kr)]/t$. (This step may be skipped if t has not changed since the last iteration.)
3. Let $r^- = k_1/(n_B - 1 - k_0)$. ($r^-$ is a lower bound for r assuming that $n_B = n_B - 1$.) If $r^- > r^+$, STOP, r^* is optimal.

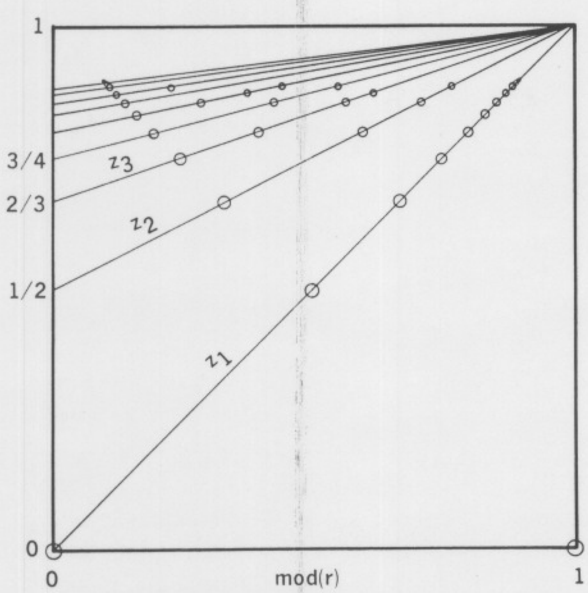


Fig. 2. $g(r)$ and linear envelopes $z_1, z_2, \ldots$.

4. Let $y = \lceil t/(1 - r^- + R^-) \rceil$ (y is the smallest integer for which $r > r^-$, given t) and let $r = R^- + (y - t)/y$. If $r \leq r^+$, set $r^* = r$ and $n_B = n_B - 1$ and go to Step 2. Otherwise let $t = t + 1$ and repeat Step 4.

Applying this algorithm to our example, it is quickly shown that with $h_A = H_A = 4$ min, $r^* = 3$, which is the integer ratio solution obtained earlier.

Experience has shown that Algorithm 1 rarely requires more than a few iterations. Experiments have also shown that there is a high likelihood that the optimal headway ratio of a randomly chosen route will be integer because of the $g(r)$ term in (5).

For the second case, the case that $h_B = H_B$, the optimal value of r lies in the range R *to* R^+. A precise algorithm for minimizing the fleet requirement under this condition is not presented, however, because in practical situations the headways h_A and h_B must usually be whole minutes (a few properties will also accept half minutes), and since $h_A = H_B/r$ in this case, there will probably be very few values of r in the range mentioned above that yield feasible values of h_A. Thus, enumeration of the feasible values of h_A below H_A, with special attention to the solution $r = R^+$, appears to be the most sensible optimization procedure.

3. MINIMIZING WAIT TIME FOR A FIXED FLEET SIZE: EXACT SOLUTION

SO FAR we have sought only to minimize the number of vehicles needed on an alternating deadheading route. Another objective is to maximize passenger level of service for a given fleet size N. The principal level of service attribute affected by a deadheading schedule is wait time, and so the objective is to minimize the sum of the headways, weighted by the ridership in the corresponding direction.

Consider first solutions in which the headway ratio r is an integer. For a given value of h_A, n_A and hence $n_B = N - n_A$ are determined. Then the minimum integer r is

$$r = \langle \pi_0/(n_B h_A - \pi_1) \rangle \tag{8}$$

where $\pi = \pi_0 + \pi_1 r$ is a linear approximation of π, given h_A and n_A.

Then finding the optimal integer solution requires only enumerating over feasible values of h_A, beginning with H_A and then trying successively decreasing acceptable values until the corresponding minimum r exceeds H_B/h_A and is therefore infeasible. Typically the number of feasible values of h_A will be very small, and experience, confirmed by the analysis presented in Section 5, suggests that in nearly all cases the optimal h_A will be H_A.

In our example problem, suppose N is fixed at 13 vehicles and we seek to minimize overall wait time. First, consider $h_A = H_A = 4$ min. Then $n_A = 11$, so $n_B = 2$. π is approximated as $9.22 + 1.77r$, and so from Equation 8 the smallest integer ratio solution is 2, implying $h_B = 8$ min. Next, consider $h_A = 3.5$ min. Then $n_A = 12$, so $n_B = 1$. π is approximated as $10.73 + 1.55r$, and so the minimum integer r is 6, which is infeasible since it leads to $h_B = 21$ min. So $h_A = 4$, $r = 2$ is the best integer ratio solution.

Relaxing the restriction of integer r, the minimum feasible r for a given h_A lies between r^* and $r^* - 1$, where r^* is the minimum integer solution. The minimum r can be found using an algorithm similar to Algorithm 1. However, the practical restriction that h_B be in whole (or in some cases half) minutes restricts the number of acceptable values of r in this range to only a few, and so enumeration of acceptable values is a simple and quick procedure.

Finally, to find the optimal number of vehicles and service headways under an objective function that includes operator cost (assumed proportional to number of vehicles) and passenger cost (assumed proportional to wait time), one need only enumerate feasible values of N beginning with the smallest feasible N and increasing by 1 each time. For each value of N, find the minimum wait time solution, and let the generalized cost for that solution be $F(N)$. (Generalized cost can include ride time as well as wait time since ride time is mildly sensitive to headway changes.) In practical situations $F(N)$ will be approximately convex, so that one should have to examine only a few values of N before the optimal solution is obvious. Then the optimal alternating deadheading schedule found may be compared with the optimal nondeadheading schedule, found using the classical square-root rule (described below) for the case of fixed demand, or using the method described in FURTH AND WILSON[5] for the case of variable demand.

4. MINIMIZING OPERATOR PLUS PASSENGER COST: APPROXIMATE SOLUTION AND ANALYTICAL INSIGHTS

IN THIS section, optimal headways are derived using some approximations that yield closed-form solutions resembling classical formulas for optimal headways. These approximate solutions can be used to reduce the amount of work done to find an optimal schedule by providing a starting point for a more exact search. They also yield insights into the nature of the optimal solution that one should expect in different situations.

For convenience, this analysis will use as the decision variable frequency instead of headway (its reciprocal). It accounts for a loading, but not a policy headway, constraint. Also for convenience, it assumes inbound to be the peak direction. The new variables used are:

- q_A = inbound frequency = $1/h_A$
- q_B = outbound frequency of in-service trips = $1/h_B$
- R_i = ridership inbound per unit time in direction i ($i = A, B$)
- c_i = ratio of peak volume to ridership in direction i
- K = vehicle capacity
- a = ratio of wait time to headway ($a = 0.5$ on a route with perfect schedule adherence)
- b = factor converting wait time to equivalent vehicle-hours (value of time).

This analysis assumes that both ridership and run time are fixed, and seeks to minimize the sum of total wait time, weighted by the factor b, and vehicle-hours.

The number of vehicle-hours of service per unit time, including only minimum necessary layover, is $t_A q_A + t_B q_B + t_D(q_A - q_B)$. This sum is a negatively biased approximation for the number of vehicles needed, since it ignores schedule slack that arises from bus integerness and from the need to keep service departures evenly spaced. This bias is smallest when there is no deadheading, since then the need to keep service departures evenly spaced does not contribute to schedule slack. Passenger wait time per unit time is $a(R_A/q_A + R_B/q_B)$. The capacity constraints take the form $q_i K \geq c_i R_i$. The problem of minimizing overall cost can thus be approximated as:

$$\min Z = t_A q_A + t_B q_B + t_D(q_A - q_B) + ab(R_A/q_A + R_B/q_B). \tag{9}$$

$$\text{s.t. } q_A \geq (c_A R_A)/K \tag{10}$$

$$q_B \geq (c_B R_B)/K \tag{11}$$

$$q_B \leq q_A \tag{12}$$

The unconstrained solution, $q_A{}^*$ and $q_B{}^*$, is

$$q_A{}^* = ((ab\ R_A)/(t_A + t_D))^{0.5} \tag{13}$$

$$q_B{}^* = ((ab\ R_B)/(t_B - t_D))^{0.5}. \tag{14}$$

Note that (13) is the familiar "square-root rule" derived by NEWELL[6] and MOHRING[7] for the isolated, fixed-demand route whose round trip time is $t_A + t_D$. It is also interesting that (14) represents an optimal headway for a route whose round trip time is only $t_B - t_D$, the deadhead premium.

Recognizing that $c_B R_B < c_A R_a$ (by virtue of A being the peak direction), the constrained solution is:

$$q_A = \max(q_A{}^*, c_A R_A/K) \tag{15}$$

$$q_B = \begin{cases} q_A & \text{if } q_B{}^* > q_A \\ \max(q_B{}^*, c_B R_B/K) & \text{otherwise.} \end{cases} \tag{16}$$

Equations 13–16 reveal some interesting insights concerning deadheading strategy in different situations. First, if the objective is to minimize operator cost only ($b = 0$), the solution is that the inbound and outbound schedules should be load-constrained.

Second, if peak direction capacity restrictions are not binding, i.e. if $q_A{}^* > c_A R_A/K$, we can show that: (a) on almost all routes, capacity restrictions will not be binding in the outbound direction either, and (b) on most routes the optimal schedule will have no deadheading.

To establish result (a), we should try to contradict the statement

$$q_A{}^* > c_A R_A/K \quad \text{and} \quad q_B{}^* < c_B R_B/K. \tag{17}$$

Multiplying the first inequality by R_B/c_A and the second by R_A/c_B, a necessary corollary of (17) is that

$$q_A{}^* R_B/c_A > q_B{}^* R_A/c_B. \tag{18}$$

After some manipulation, (18) becomes

$$(t_B - t_D)/(t_A + t_D) > (R_A/R_B)(c_B/c_A)^2. \quad (19)$$

However, R_A/R_B will be considerably greater than 1 (since A is the peak direction); c_B/c_A will usually be close to 1; and $(t_B - t_D)$, the deadhead premium, will usually be far less than $(t_A + t_D)$, the round trip time of a deadheading vehicle, so that (19) will not hold under common circumstances, implying a contradiction.

Thus we have shown that under usual circumstances, when the peak direction schedule is not load-constrained, the reverse direction schedule will not be load constrained either, so that (16) becomes

$$q_B = \min(q_B^*, q_A^*). \quad (20)$$

To establish result (b), we notice from (20) that there will be no deadheading unless $q_B^* < q_A^*$. With some manipulation, this condition can be expressed as

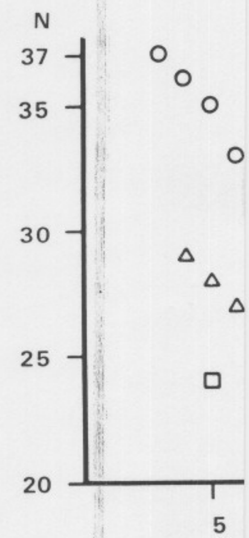
$$(t_B - t_D)/(t_A + t_B) > R_B/(R_A + R_B). \quad (21)$$

Recognizing that this approximate approach overestimates the benefits of deadheading, (21) can be interpreted as stating that if the peak direction headway is not load-constrained, there should be no deadheading in the reverse direction unless the deadhead premium, as a fraction of the in-service round trip time, considerably exceeds the reverse direction's share of the route's total patronage. This condition is one that will not hold on most local routes. Hence, on most routes, the optimal alternating deadheading schedule, if it includes any deadheading at all, will have $h_A = H_A$.

5. APPLICATION

THESE DESIGN methods were applied to San Francisco Municipal Railway Route 14, a trolley bus route which runs 9 miles along Mission Street from the southern boundary of San Francisco to the Ferry Terminal downtown. The route is paralleled by the I-280 freeway, making it a good candidate for alternating deadheading if diesel buses were used. For the sake of illustration, the data have been somewhat simplified. During the a.m. peak, the route operates at 4-min headways in each direction, with running times (including minimum necessary layover) of 56 min inbound and 58 min outbound. It was assumed that the increase in running time resulting from a 1-min increase in headway was 1 min in the outbound direction and 2 min in the inbound direction. The deadheading time outbound via I-280 was estimated to be 30 min, including necessary layover.

Figure 3 shows the fleet size required with different inbound and



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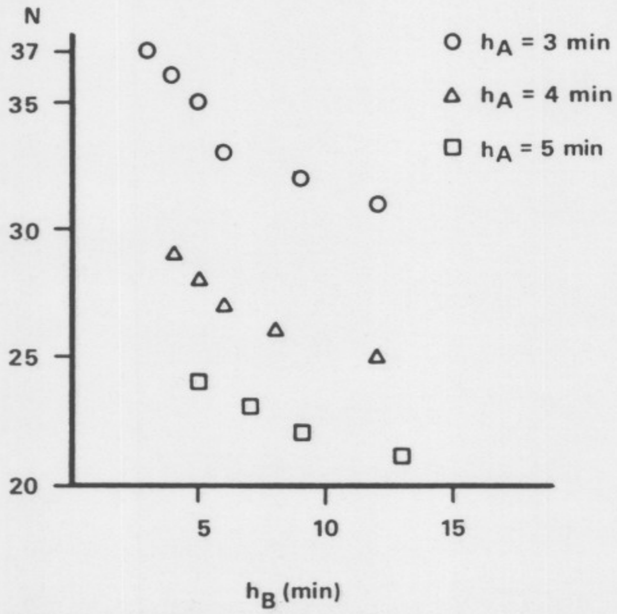

Fig. 3. Fleet size required for different alternating deadheading schedules.

outbound service headways. Inbound headways of 3, 4, and 5 min were examined. The only values of h_B indicated in Figure 3 are the "superior solutions"—the minimum feasible h_B for a given N and h_A. These results show that significant vehicle savings are possible through alternating deadheading. At the existing inbound headway of 4 min, 29 buses are needed if there is no deadheading ($h_B = 4$). If the outbound peak passenger volume is 2/3 of the inbound peak volume, implying that h_B could be raised to 6 min, then 2 out of 29 buses could be saved, a savings of 6.9%; and if the outbound peak volume were half the inbound peak volume, implying that h_B could be 8 min, then 3 out of 29 buses could be saved, a savings of 10.3%.

Alternating deadheading was also incorporated in a corridor route design study that considered the complementary design of zonal local and express bus routes (FURTH[8]). It was assumed that express routes would completely deadhead, while local routes were allowed to use an alternating deadheading schedule. Algorithms for alternating deadheading design were embedded in a larger design routine that chose market segments for each service type and service zones for each local and express route as well as headway. Depending on the objective, alternating deadheading schedules for the local routes were sometimes included in the optimal design.

6. CONCLUSIONS

THE WORK reported in this paper provides the transit planner with methods for computing the fleet requirements of an alternating deadheading schedule, and for finding the best alternating deadheading schedule for a local route under the usual scheduling constraints. These methods can be executed manually in a short amount of time, and the more complex algorithms are very quickly performed on a computer. As a strategy, alternating deadheading can reduce moderately the number of vehicles needed on many existing routes, and should be incorporated in the design of bus service in any high demand corridor.

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